Bessel's function:- The differential equation,

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$$

or,

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right)y = 0$$

is known as Bessel's differential equation.

The Solution of the differential equation is given by,

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r \, \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

Putting $-n$ for n, we get

$$J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r \, \Gamma(r+1-n)} \left(\frac{x}{2}\right)^{-n+2r}$$

$$y = A\,J_n(x) + B\,J_{-n}(x).$$

① QN.→ Prove that,

$$J_{-n}(x) = (-1)^n J_n(x).$$

Solu^n. We have,

$$J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r \, \Gamma(r+1-n)} \left(\frac{x}{2}\right)^{-n+2r}$$

If $r = 0, 1, \ldots\ldots n-1$, $\Gamma(r+1-n)$ has infinite value which means that the terms corres-

ponding to these values vanish,
So, we have

$$J_{-n}(x) = \sum_{r=n}^{\infty} \frac{(-1)^r}{\lfloor r \, \Gamma(r+1-n)} \left(\frac{x}{2}\right)^{-n+2r}$$

Putting $r = n+s$, we get

$$J_{-n}(x) = \sum_{s=0}^{\infty} \frac{(-1)^{n+s}}{\lfloor n+s \cdot \Gamma(s+1)} \left(\frac{x}{2}\right)^{n+2s}$$

Also, $\lfloor n+s = \Gamma(n+s+1)$ and $\Gamma(s+1) = \lfloor s$.

$$\text{So, } J_{-n}(x) = \sum_{s=0}^{\infty} \frac{(-1)^n \cdot (-1)^s}{\lfloor s \cdot \Gamma(n+s+1)} \left(\frac{x}{2}\right)^{n+2s}$$

$$= (-1)^n J_n(x).$$

② Q.No.→ Prove that

$$\frac{d}{dx}\left(J_0(x)\right) = -J_1(x).$$

Soln We have,

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r \, \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

Putting $n=0$, we get

$$J_0(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r \cdot \lfloor r} \left(\frac{x}{2}\right)^{2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{(\lfloor r)^2} \left(\frac{x}{2}\right)^{2r}$$

$$= 1 - \left(\frac{x}{2}\right)^2 + \frac{1}{(\lfloor 2)^2}\left(\frac{x}{2}\right)^4 - \frac{1}{(\lfloor 3)^2}\left(\frac{x}{2}\right)^6 + \cdots$$

$$\therefore \frac{d}{dx}\left(J_0(x)\right) = -2\left(\frac{x}{2}\right)\cdot\frac{1}{2} + 4\cdot\frac{1}{(\lfloor 2)^2}\left(\frac{x}{2}\right)^3\cdot\frac{1}{2}$$

$$- \overset{3}{\cancel{6}}\,\frac{1}{(\lfloor 3)^2}\left(\frac{x}{2}\right)^5\cdot\frac{1}{\cancel{2}} + \cdots$$

$$= -\frac{x}{2} + \frac{1}{\lfloor 1\ \lfloor 2}\left(\frac{x}{2}\right)^3 - \frac{1}{\lfloor 2\ \lfloor 3}\left(\frac{x}{2}\right)^5 + \cdots$$

$$= -\sum_{r=0}^{\infty}(-1)^r \frac{1}{\lfloor r\ \lfloor r+1}\left(\frac{x}{2}\right)^{1+2r}$$

$$= -\sum_{r=0}^{\infty}(-1)^r \frac{1}{\lfloor r\ \Gamma(r+1+1)}\left(\frac{x}{2}\right)^{1+2r}$$

$$= -J_1(x).$$

M.Sc. Q.No.→ Prove the recurrence relation for Bessel functions $J_n(x)$,

$$x J_n'(x) = n J_n(x) - x J_{n+1}(x).$$

or, Q.No.→ $J_n'(x) = \frac{n}{x} J_n(x) - J_{n+1}(x)$

OR, $\frac{d}{dx} J_n(x) = \frac{n}{x} J_n(x) - J_{n+1}(x)$.

OR (I) $x J_n' = n J_n - x J_{n+1}$.

Proof:- We have,

$$J_n = \sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r \, \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

Differentiating w.r.t. x, we get

$$J_n' = \sum_{r=0}^{\infty} \frac{(-1)^r (n+2r)}{\lfloor r \cdot \Gamma(n+r+1)} \cdot \frac{1}{2} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= n \cdot \sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r \, \Gamma(n+r+1)} \cdot \left(\frac{x}{2}\right)^{n+2r} \frac{\cancel{2}}{x} \cdot \frac{1}{\cancel{2}}$$

$$+ \sum_{r=0}^{\infty} \frac{(-1)^r \cancel{2} r}{\lfloor r \, \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1} \cdot \frac{1}{\cancel{2}}$$

or, $x J_n' = n \cdot J_n + x \sum_{r=1}^{\infty} \frac{(-1)^r \cdot r}{\lfloor r \, \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1}$

$$= n \cdot J_n + x \cdot \sum_{r=1}^{\infty} \frac{(-1)^r}{\lfloor r-1 \, \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1}$$

Putting $(r-1) = s$ in the above summation, we get

$$\therefore \; xJ_n' = nJ_n + x\sum_{s=0}^{\infty} \frac{(-1)^{s+1}}{\lfloor s \, \Gamma(n+s+2)} \left(\frac{x}{2}\right)^{n+2s+1}$$

$$= nJ_n - x\sum_{s=0}^{\infty} \frac{(-1)^{s}}{\lfloor s \, \Gamma(n+1+s+1)} \left(\frac{x}{2}\right)^{(n+1)+2s}$$

$$xJ_n' = nJ_n - xJ_{n+1}$$

i.e. $xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x)$

or, $J_n'(x) = \frac{n}{x}J_n(x) - J_{n+1}(x)$

~~or,~~ $\cancel{\frac{d}{dx}J_n(x) = \frac{n}{x}J_n(x)}$

or, $\frac{d}{dx}J_n(x) = \frac{n}{x}J_n(x) - J_{n+1}(x)$

Q.No (II) $xJ_n' = -nJ_n + xJ_{n-1}$.

or, $J_n'(x) = J_{n-1}(x) - \frac{n}{x} J_n(x)$.

Soln: We have

$$J_n = \sum_{r=0}^{\infty} \frac{(-1)^r}{\underline{|r}\; \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

Differentiating both sides, w.r.t. x, we get

$$J_n' = \sum_{r=0}^{\infty} \frac{(-1)^r (n+2r)}{\underline{|r}\; \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1} \cdot \frac{1}{2}$$

$$\text{or, } xJ_n' = \sum_{r=0}^{\infty} \frac{(-1)^r (n+2r)}{\underline{|r}\; \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r (2n+2r-n)}{\underline{|r}\; \Gamma(n+r+1)} \cdot \left(\frac{x}{2}\right)^{n+2r}$$

$$= \sum_{r=0}^{\infty} \frac{2(n+r)(-1)^r}{\underline{|r}\; (n+r)\, \Gamma(n+r)} \left(\frac{x}{2}\right)^{n+2r} - n \sum_{r=0}^{\infty} \frac{(-1)^r}{\underline{|r}\; \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= -nJ_n + \sum_{r=0}^{\infty} \frac{(-1)^r \cdot \cancel{2}}{\underline{|r} \cdot \Gamma(n+r)} \cdot \frac{x}{\cancel{2}} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= -nJ_n + x \sum_{r=0}^{\infty} \frac{(-1)^r}{\underline{|r} \cdot \Gamma(n-1+r+1)} \left(\frac{x}{2}\right)^{n-1+2r}$$

$$xJ_n' = -nJ_n + xJ_{n-1}.$$

i.e. $xJ_n'(x) = -nJ_n(x) + xJ_{n-1}(x)$

or, $J_n'(x) = J_{n-1}(x) - \frac{n}{x} J_n(x)$.